IMSC 2058 Solution Midterm

1. (i) Let $S_n = \{(x_1, x_2, \cdots, x_n, 0, \cdots, 0) \in \mathbb{R}^\infty : x_k = 0 \ \forall k > n\}$. Then S_n is isomorphic to $\mathbb{R}^n$.To show $\mathbb{R}^n$ is a nowhere dense subset of $(\mathbb{R}^\infty, d)$ is equivalent to show the closure $\overline{S_n}$ has no interior points.

   Suppose $\{x^{(m)}\}_{m=1}^\infty \subseteq S_n$ with $x^{(m)} \to x$ in $(\mathbb{R}^\infty, d)$, so $d(x^{(m)}, x) \to 0$. For any fixed $k > n$, we have $x_k^{(m)} = 0$ for all m, so $|x_k^{(m)} - x_k| = |x_k| \to 0$ implies $x_k = 0$. Thus, $x \in S_n$, so S_n is closed, i.e. $S_n = \overline{S_n}$.

   Let $y \in S_n$ and $r > 0$ be arbitrary. Consider the open ball $B(y, r) = \{z \in \mathbb{R}^\infty : d(y, z) < r\}$. Define $z \in \mathbb{R}^\infty$ by $z_k = y_k$ for $1 \le k \le n$, $z_{n+1} = r/2$, and $z_k = 0$ for $k > n+1$. Then $z \notin S_n$ (since $z_{n+1} \neq 0$), but

$$d(y, z) = \sqrt{\sum_{k=1}^{\infty} (z_k - y_k)^2} = \sqrt{(r/2)^2} = r/2 < r,$$

   so $z \in B(y, r)$. Thus, no open ball around y is contained in S_n, so $\operatorname{int}(S_n) = \emptyset$.

   Therefore, S_n is a nowhere dense subset of $(\mathbb{R}^\infty, d)$ which implies $\mathbb{R}^n$ is also a nowhere dense subset of $(\mathbb{R}^\infty, d)$.

   (ii) No.

   The Baire Category Theorem states that a complete metric space cannot be written as a countable union of nowhere dense sets. Since $\mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$ is such a union, $(\mathbb{R}^\infty, d)$ is not complete.

   (iii) No.

   Consider the bounded sequence $\{e_k\}_{k=1}^\infty$ where e_k is the standard basis vector with 1 in the k-th position and 0 elsewhere. It is bounded, but has no convergent subsequence: for any subsequence $\{e_{k_\ell}\}$, $d(e_{k_\ell}, e_{k_m}) = \sqrt{2}$ for $\ell \neq m$, so it is not Cauchy hence does not converge.

2. (i) τ is the discrete topology if and only if every singleton set $\{x\} \in \tau$ for all $x \in X$.

   Note that a basis for the product topology τ on $X = \prod_{n=1}^\infty X_n$ consists of sets of the form $\prod_{n=1}^\infty U_n$, where U_n is an open set in X_n for each n, and $U_n = X_n$ for all but a finite number of indices.

   Let $x = (x_1, x_2, \dots)$ be an arbitrary point in X. If $\{x\}$ is an open subset of X, there exists $\prod_{n=1}^\infty U'_n$ in τ, such that $\{x\} = \prod_{n=1}^\infty U'_n$. It follows that $U'_n = \{x_n\}$ for all n, and $\prod_{n=1}^\infty U'_n \notin \tau$, which gives a contradiction. Therefore τ is not a discrete topology.

   (ii) Define a metric d on X by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n),$$

   where ρ_n is the given discrete metric on X_n. This is well-defined since $0 \le \rho_n(x_n, y_n) \le 1$ for all n and $\sum_{n=1}^\infty \frac{1}{2^n} = 1$. Then we check that d is a metric:

   - $d(x, y) \ge 0$ for all $x, y \in X$, because $\rho_n(x_n, y_n) \ge 0$.

- $d(x,y)=0$ if and only if $x_n = y_n$ for all n, which implies $x=y$.
- Since $\rho_n(x_n,y_n)=\rho_n(y_n,x_n)$ then $d(x,y)=d(y,x)$ for all $x,y\in X$.
- The Triangle Inequality of the $\rho_n(x_n,y_n)$ induces $d(x,y)\le d(x,z)+d(z,y)$ for all $x,y,z\in X$.

Fix $x=(x_1,x_2,\dots)\in X$ and $r>0$; choose $k\in\mathbb{N}$ such that $\sum_{n=k+1}^{\infty}1/2^n<r/2$. Let $B_d(x,r)$ denote the open ball center at x with radius r.

Let $V=\prod_n U_n$ where $U_n=\{x_n\}$ for $n\le k$ and $U_n=X_n$ for $n\ge k+1$. if $y\in V$, then $d(x,y)\le\sum_{n=k+1}^{\infty}1/2^n<r/2<r$. Hence $V\subseteq B_d(x,r)$.

Conversely, let a basic open set W in the product topology τ takes the form $W=\prod_{n=1}^{\infty}U_n$, where each U_n is open in the discrete space X_n and $U_n=X_n$ for all but finitely many n, and $W\neq\emptyset$. Without loss of generality, assume the finite set of indices where $U_n\neq X_n$ is $F=\{m_1,\dots,m\}\subseteq\mathbb{N}$ with $m_1<\cdots<m$. Let $r'=\frac{1}{2^{m+1}}$ Fix $z\in W$. Now claim: $B_d(z,r')=\{y\in X: d(z,y)<r'\}\subseteq W$.

Let $y\in B_d(z,r')$. Suppose for contradiction that $y\notin W$. Then there exists some $m_i\in F$ with $y_{m_i}\notin U_{m_i}$, so $y_{m_i}\neq z_{m_i}$ and thus $\rho_{m_i}(z_{m_i},y_{m_i})=1$. For this fixed m_i, the sum splits as

$$d(x,y)=\sum_{n\neq m_i}\frac{1}{2^n}\rho_n(x_n,y_n)+\frac{1}{2^{m_i}}\cdot 1\ge\frac{1}{2^{m_i}}>r',$$

which contradict with $y\in B_d(z,r')$. Hence $B_d(z,r')\subseteq W$ and the metric topology induced by d equals τ.

3. (i) Assume $A\cap B=\emptyset$. Then $B\subseteq\mathbb{R}\setminus A$. Since A is open dense subsets of $\mathbb{R}$, then the set $\mathbb{R}\setminus A$ is closed and has empty interior.

   However, B is open and dense, so B contains some nonempty open interval $U\subseteq B$. This U is a nonempty open set contained in $\mathbb{R}\setminus A$, contradicting the fact that $\mathbb{R}\setminus A$ has empty interior. Thus, the assumption is false, and $A\cap B\neq\emptyset$.

   (ii) Let $\overline{x}\neq\overline{y}$ in $\mathbb{R}/\mathbb{Z}$, so $x-y\notin\mathbb{Z}$. Without loss of generality, assume $0\le\{x\}<\{y\}<1$, where $\{\cdot\}$ denotes fractional part (If $x=5.432$ then $\{x\}=0.432$). Choose $\epsilon>0$ small enough that the open intervals $U=(x-\epsilon,x+\epsilon)$ and $V=(y-\epsilon,y+\epsilon)$ are disjoint in $\mathbb{R}$. Let $U'=\bigcup_{n\in\mathbb{Z}}(x+n-\epsilon,x+n+\epsilon)$ and $V'=\bigcup_{n\in\mathbb{Z}}(y+n-\epsilon,y+n+\epsilon)$; both are open in $\mathbb{R}$. Thus, $\pi(U')$ and $\pi(V')$ are disjoint open neighborhoods of $\overline{x}$ and $\overline{y}$ in $\mathbb{R}/\mathbb{Z}$. It follows that the quotient $\mathbb{R}/\mathbb{Z}$ is Hausdorff.

   Since the closed interval $[0,1]$ is compact in $\mathbb{R}$. We restrict the projection π on $[0,1]$ yield

   $$\pi|_{[0,1]}:[0,1]\to\mathbb{R}/\mathbb{Z}$$

   It is continuous and surjective. Therefore $\mathbb{R}/\mathbb{Z}\cong\pi([0,1])$ is compact.

   (iii) The quotient group $\mathbb{R}/\mathbb{Q}$ is not Hausdorff under the quotient topology.

   Let $a=\overline{0}=\mathbb{Q}$ and $b=\overline{\sqrt{2}}=\sqrt{2}+\mathbb{Q}$. These are distinct since $\sqrt{2}\notin\mathbb{Q}$. Suppose, for contradiction, there exist disjoint open sets $U,V\subset\mathbb{R}/\mathbb{Q}$ with $a\in U$ and $b\in V$. Then $W=\pi^{-1}(U)$ and $Z=\pi^{-1}(V)$ are disjoint open sets in $\mathbb{R}$ satisfying $W\supset a=\mathbb{Q}$ and $Z\supset b=\sqrt{2}+\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore W and Z are open and dense in $\mathbb{R}$. It follows from 3(i) that $W\bigcap Z\neq\emptyset$ which gives contradiction. Therefore, no such disjoint open U and V exist, so $\mathbb{R}/\mathbb{Q}$ is not Hausdorff.